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MEMORANDUM

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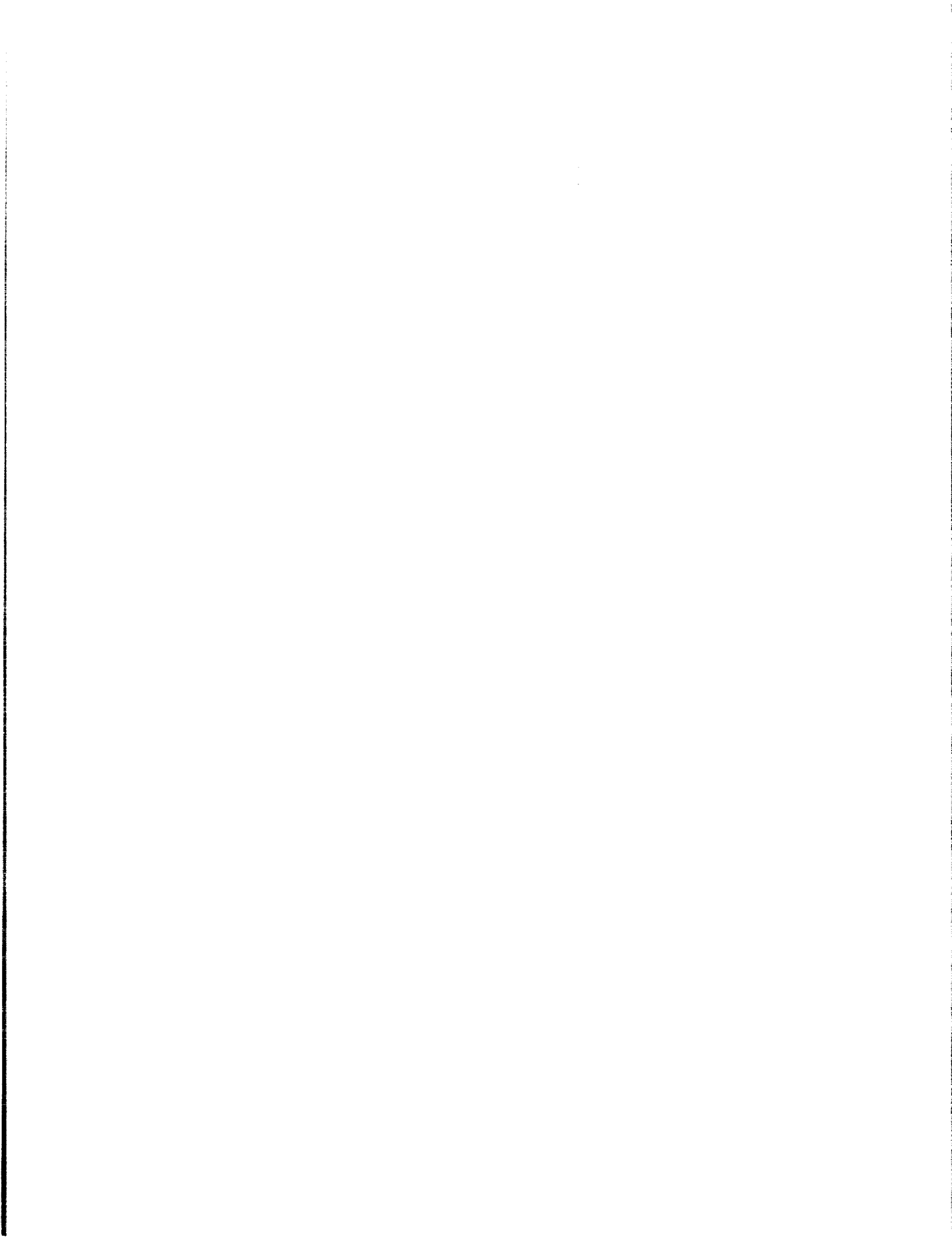
VARIABLE VELOCITY

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VARIABLE VELOCITY

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SUMMARY

The problem considered is that of rectilinear motion with variable velocity. The paper gives, by an elementary construction, a system of coordinates which is conformal in the vicinity of the axis of motion. By a particular choice of the scale relation, such restricted conformal transformations can be made to reduce to the Lorentz transformation everywhere in the case of uniform velocity and locally in the case of variable velocity.

INTRODUCTION

The use of Lorentz transformations when the velocity v is a function of the time leads to a nonuniform correspondence of events between accelerated coordinate systems. A uniform correspondence can be achieved, however, by a simple generalization of the Lorentz formula to a transformation of the type known as conformal.

Extension of the theory of relativity by conformal transformations was considered many years ago by H. Bateman (ref. 1). Such transformations preserve a constant velocity of light even during accelerated motions. It seems, however, that such transformations, if applied to the whole of space, are consistent only with special types of motion. In the present paper an extension of the theory by conformal transformations of the simplest possible type, restricted to the vicinity of the axis of motion, is suggested. Such restricted conformal transformations can be made to reduce to the Lorentz transformation everywhere in the case of constant velocity, and locally in the case of variable velocity.

The paper gives an elementary (though somewhat indirect) derivation of the coordinates from the wave pattern formed by electromagnetic signals. In view of the simplicity of the results it seemed unnecessary to translate them into the more concise language of differential geometry.

CONSTRUCTION OF A CONFORMAL COORDINATE SYSTEM
BY MEANS OF INTERFERING WAVE SIGNALS

In the Michelson-Morley experiment, an electromagnetic oscillator and a mirror are arranged on a rigid reference body B so as to produce an interfering wave system. Since the waves returned by the mirror are of the same frequency as those emitted by the oscillator, a system of standing waves results.

We may produce the same wave system by means of two oscillators at x_p and x_q on the x axis in an A system which moves relatively to B. Here the waves returned by the upstream oscillator may have a different frequency than those emitted by the downstream oscillator. The inter-ference pattern will then drift in the direction of the line joining the two oscillators. Such an interference pattern moving at the velocity v is of course a standing wave in the B system. The difference in frequency is then related to the Doppler shift as observed from system A.

If we imagine Einstein's clock synchronization experiments (ref. 2) to be carried out continuously, then these experiments will again create a standing wave system in which the phase of a clock is identified with the phase of the electromagnetic wave and the length of a measuring rod (or the spacing between the clocks) is identified with the wave length.

Figure 1 shows an x,t diagram of the clock experiments interpreted as an interfering wave pattern. For convenience the x and t scales are chosen so that the velocity of light is unity. The rectangular x,t axes are those of system A. Maxima and minima of the downstream moving waves are along lines sloping to the left with increasing t, or along lines x + t = constant. The upstream moving waves are identified similarly with the lines x - t = constant.

The maxima and minima of the standing waves in the B system are diagonals of the rectangles formed by the lines x - t = constant and x + t = constant. The Lorentz transformation is, in these terms,

$$x' + t' = \sqrt{\frac{1-v}{1+v}}\,(x+t)$$

$$x' - t' = \sqrt{\frac{1+v}{1-v}}\,(x-t)$$

$$\tag{1}$$

In system B (x',t') we may isolate two clocks C_1 and C_2. A signal from C_1 at t_1' arrives at C_2 at the time t_2' and is reflected

back to C_1, arriving there at the time t_3'. By setting C_2 to the time $t_2' = \frac{1}{2}(t_1' + t_3')$ we express the constancy of the velocity of light in the moving system.

Now it is clear that the velocity of light will remain constant and that the Michelson-Morley experiment would show this result in any coordinate system constructed from interfering waves of both families. Hence as the equation of such a transformation we may write, disregarding for the moment y and z,

$$x' + t' = F(x+t)$$
$$x' - t' = G(x-t) \qquad (2)$$

Figure 2 shows a curvilinear coordinate system constructed in this way. By starting with the Doppler variation of the frequencies received at the points x_p and x_q we obtain a network of lines at 45°, but with variable spacing. The interference pattern then forms a family of "sloshing" waves with the velocity of the wave crests variable from point to point. In a coordinate system tied to such wave crests the result of the Michelson-Morley experiment is always the same. It is a simple matter to verify that the coordinate system of figure 2 preserves a constant velocity of light.

If we denote

$$F'(x+t) = f$$
$$G'(x-t) = g \qquad (3)$$

then the velocity v of B relative to A is given by

$$\left(\frac{dx}{dt}\right)_{x'=\text{const}} = v = \frac{g-f}{g+f} \qquad (4)$$

satisfied. Evidently Milne's relation cannot be maintained by conformal transformations that are locally tangent to the Lorentz transformation, except in the case of uniform velocity.

Figure 3 illustrates the application of the present method to a problem of accelerated rectilinear motion. If the path of a particle B is given then it is only necessary to lay off from this path lines at 45°, spaced at equal intervals of the proper time as given by the velocity v in conjunction with the restricted theory of relativity. The intersections of these characteristic lines then determine the continuation of the coordinates throughout the x,t region.

Thus far our considerations have been restricted to the single space direction x parallel to the direction of the velocity. If the constancy of the velocity of light is to be maintained in all directions, then the differential form of the transformation must remain.

$$-ds'^2 = \lambda^2(x,y,z,t)(dx^2+dy^2+dz^2-dt^2) \tag{10}$$

Generalization of the theory of relativity by transformations of the group satisfying equation (10) was considered by Bateman (ref. 1) and more recently by Infeld and Schild (ref. 4), Littlewood (ref. 5), E. L. Hill (ref. 6), and others. It seems that the transformations of this group are essentially limited in such a way that only certain motions are consistent with a constant velocity of light throughout space. However, if we restrict our attention to a small region in the vicinity of the x axis, we may write

$$\left. \begin{array}{l} y' = \sqrt{fg}\; y \\ z' = \sqrt{fg}\; z \end{array} \right\} \tag{11}$$

and then

$$\left(-ds'^2\right)_{y^2+z^2 \to 0} = fg(dx^2+dy^2+dz^2-dt^2) \tag{12}$$

Hence a constant velocity of light can be maintained in the vicinity of the axis throughout a variable motion.

It is interesting to note (see fig. 3) that the coordinate distortions or "gravitational waves" associated with the acceleration of system B ultimately propagate away from the origin with the velocity of light. The interpretation of these waves, as well as other dynamical questions, will require an extension of the considerations given herein.

Ames Research Center
National Aeronautics and Space Administration
Moffett Field, Calif., Apr. 9, 1959

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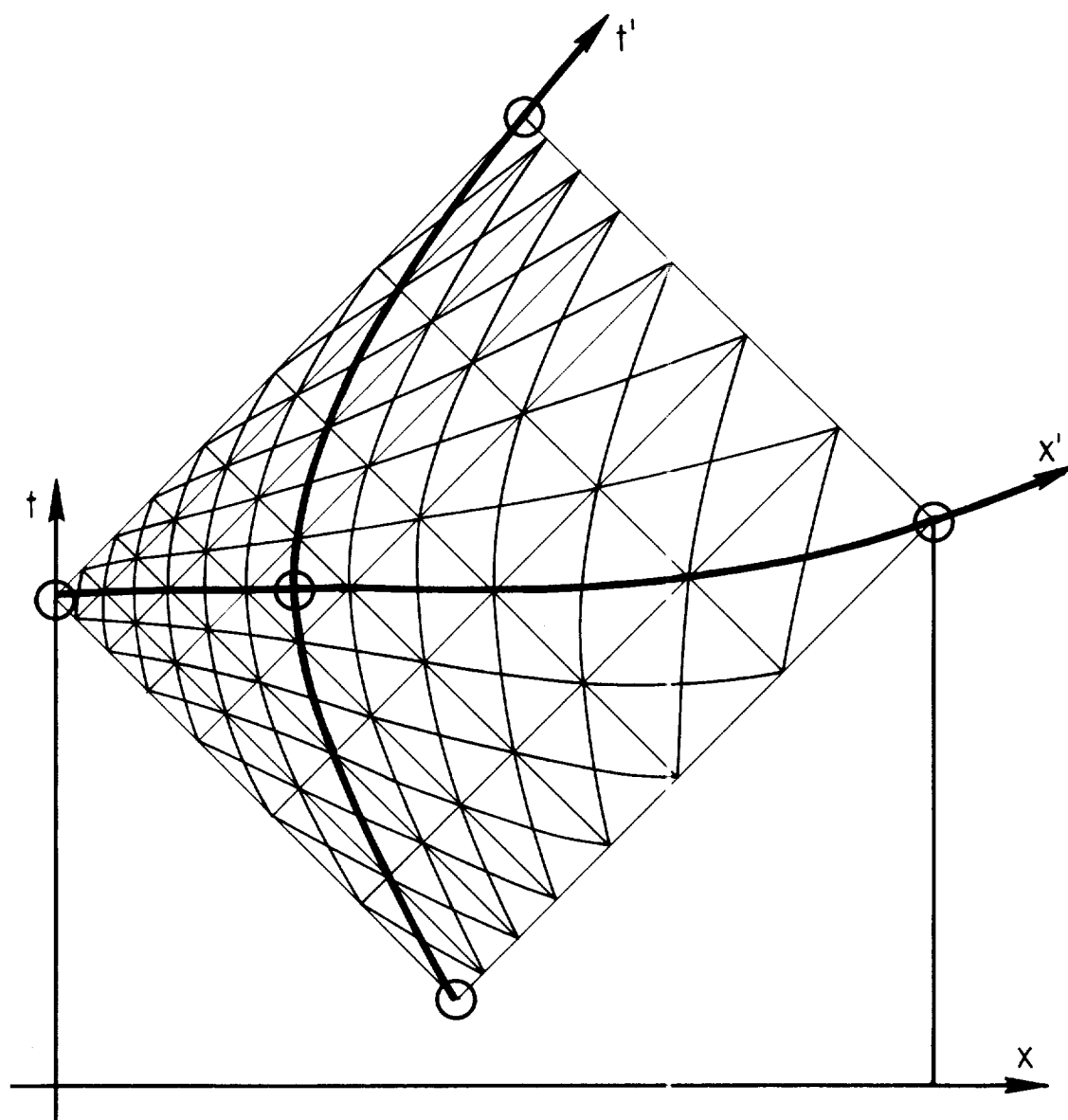
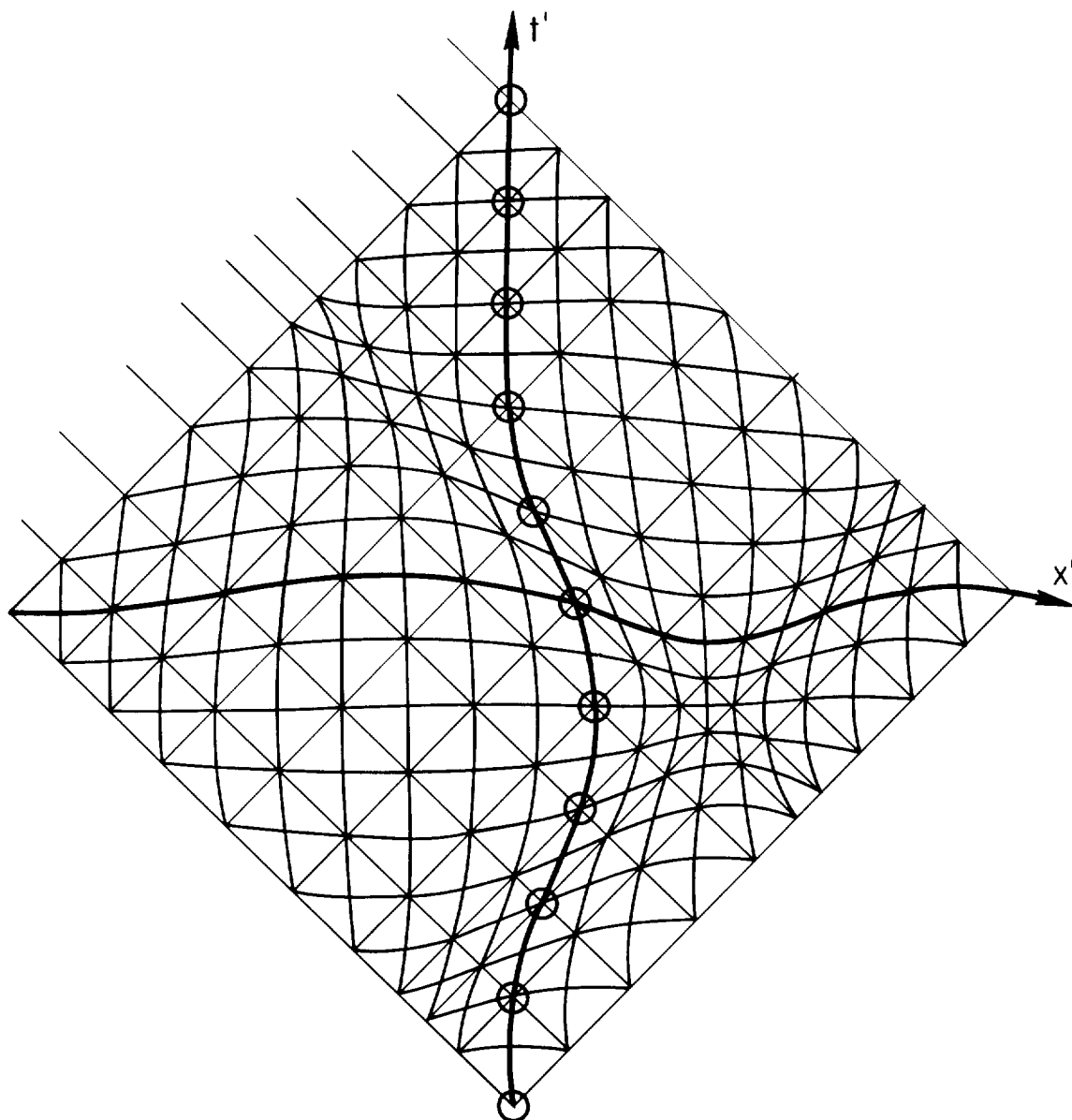Figure 2.- Coordinates in which the velocity of light is constant.

Figure 3.- Continuation of Lorentz transformation along straight
characteristic lines.